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Algorithms for finding connected separators between antipodal points

Jan P. Boroński [a], Piotr Minc [a], Marian Turzański [b,*]

[a] *Department of Mathematics and Statistics, Auburn University, Alabama 36849, USA*
[b] *Faculty of Mathematics and Natural Sciences, College of Sciences, Cardinal Stefan Wyszyǹski University, ul. Dewajtis 5, 01-815 Warszawa, Poland*

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Abstract

A set (or a collection of sets) contained in the Euclidean space $\mathbb{R}^m$ is symmetric if it is invariant under the antipodal map. Given a symmetric unicoherent polyhedron X (like an n-dimensional cube or a sphere) and an odd real function f defined on vertices of a certain symmetric triangulation of X, we algorithmically construct a connected symmetric separator of X by choosing a subcollection of the triangulation. Each element of the subcollection contains the vertices v and u such that $f(v)f(u) \leqslant 0$.

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1. Introduction

In this paper, we use the following notation and definitions. For any collection $\mathcal{K}$, by $\mathcal{K}^*$ we denote the union of elements of $\mathcal{K}$. $\mathbb{R}$ denotes the real line, I denotes the closed segment $[-1,1]$ contained in $\mathbb{R}$. For any positive integer n, S^n denotes the unit sphere in $\mathbb{R}^{n+1}$. By the *antipodal map* defined on a set $Z \subset \mathbb{R}^m$, where m is a positive integer, we understand the map assigning $-z$ to each $z \in Z$. By $-Z$ we denote the image of Z under the antipodal map. A set (or a collection of sets) contained in $\mathbb{R}^m$ is *symmetric* if it is invariant under the antipodal map. Clearly, I^n and S^n are symmetric. Let $\mathbf{0}$ denote the origin of $\mathbb{R}^m$. Clearly, $\mathbf{0}$ is the only fixed point of the antipodal map on $\mathbb{R}^m$.

For a symmetric set Z and its symmetric closed subset C, we say that C is a *symmetric separator* of Z if C separates Z between z and $-z$ for each point $z \in Z \setminus C$.

For a symmetric set Z and a function $g: Z \to \mathbb{R}^k$, we say that g is *odd* if $g(-z) = -g(z)$ for each $z \in Z$. For a function $f: Z \to \mathbb{R}^k$, let $s[f]: Z \to \mathbb{R}^k$ be defined by $s[f](z) = f(z) - f(-z)$. Observe that $s[f]$ is odd, and it is continuous if f is continuous.

* Corresponding author.
E-mail addresses: boronjp@auburn.edu (J.P. Boroński), mincpio@auburn.edu (P. Minc), mtturz@ux2.math.us.edu.pl (M. Turzański).

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If g is an odd real–valued and continuous map defined on a symmetric set Z, then $g^{-1}(0)$ is a symmetric separator of Z. Conversely, if C is a symmetric separator of a locally connected compact set Z, one can easily construct a map $g_C : Z \to \mathbb{R}$ such that ${g_C}^{-1}(0) = C$. (Observe that since Z is locally connected compact set, each component of $Z \setminus C$ is open in Z. Arrange components of $Z \setminus C$ into a sequence $(Z_1, Z_2, \ldots)$. Let U be the union of those components Z_i such that $-Z_i = Z_j$ for some $j > i$. Observe that U and $-U$ are open, $-U \cap U = \emptyset$ and $-U \cup U = Z \setminus C$. For $z \in U \cup C$, define $g_C(z)$ to be the distance of z from C. Set $g_C(z) = -g_C(-z)$ for $z \in -U$.)

Suppose that T is a homeomorphism of a space X onto itself. We say that T is an involution on X if T^2 is the identity on X. The antipodal map restricted to any symmetric set X is an involution on X.

It is convenient to generalize the notion of odd functions by replacing the antipodal map by an arbitrary involution. Suppose that T is an involution on a space X. We say that a set $Z \subset X$ is *T-symmetric* if $T(Z) = Z$. We say $C \subset X$ is a *T-symmetric separator* of X if C separates X between x and $T(x)$ for each point $x \in X \setminus C$. A real function g defined on a T symmetric set Z is a *T-odd function* if $g(T(z)) = -g(z)$ for each $z \in Z$.

By a continuum we understand a connected and compact metric space. A continuum X is *unicoherent* if $A \cap B$ is connected for any subcontinua A and B of X such that $A \cup B = X$. Each cube I^n is unicoherent. The circle S^1 is not unicoherent. But, for each $n \geqslant 2$, the sphere S^n is unicoherent.

A space X has the *fixed point property* if for each continuous map $f : X \to X$ there is a point $x \in X$ such that $f(x) = x$. In 1911, Brouwer [2] proved that the n-dimensional cube I^n has the fixed point property. The n dimensional sphere does not have the fixed point property. For instance, since $\mathbf{0} \notin S^n$, the antipodal map has no fixed point on S^n. However, the sphere satisfies the following coincidence theorem.

Theorem. *(Borsuk–Ulam theorem on antipodes, 1933 [3].) For any map $f : S^n \to \mathbb{R}^n$ there exists a point $x \in S^n$ such that $f(x) = f(-x)$.*

The above theorem can be restated in terms of odd maps and symmetric separators.

Restatement. *Each of the following two conditions is equivalent to the Borsuk–Ulam theorem.*

(1) *For any odd map $g : S^n \to \mathbb{R}^n$ there exists a point $x \in S^n$ such that $g(x) = \mathbf{0}$.*
(2) *The intersection of any n symmetric separators of S^n is not empty.*

Indeed, (1) follows directly from the theorem. We get the theorem from (1) by using the condition with $g = s[f]$. We get (1) from (2) by taking $g = (g_1, \ldots, g_n)$ and $x \in \bigcap_{i=1}^n {g_i}^{-1}(0)$. Finally, assume that (1) is true and $C_1, C_2, \ldots, C_n$ are symmetric separators of S^n. Now, use (1) with $g = (g_{C_1}, g_{C_2}, \ldots, g_{C_n})$ to get $x \in S^n$ such that $g(x) = \mathbf{0}$. Observe that $x \in \bigcap_{i=1}^n C_i$, so the intersection of $C_1, C_2, \ldots, C_n$ is not empty and (2) is true.

It should be noted that we may consider only connected symmetric separators in (2). In fact, Floyd [7] proved in 1955 that if T is a fixed-point-free involution on a locally unicoherent continuum X, then each T-symmetric separator of X contains a connected T-symmetric separator of X. A similar result was obtained independently by Haman and Kuratowski in [12]. See also [11,13,15] for related results and generalizations.

Brouwer fixed point theorem and Borsuk–Ulam theorem on antipodes are closely related. A proof of the former may be obtained as a simple corollary of the latter. Both of the theorems play important role in mathematics and have numerous applications. Both theorems have been objects of intense research. There are many different proofs of each of them. Some of those proofs are constructive and lead to algorithmic techniques that could be used numerically. In 1967 Scarf in [18] presented an algorithm to approximate fixed points promised by the Brouwer fixed-point theorem. Constructive proofs for Borsuk–Ulam theorem were given by Alexander and Yorke in [1], and Meyerson and Wright in [17]. (See also [6,4,19,21] for related results.)

In this paper we give an algorithm to approximate a connected symmetric separator contained in $g^{-1}(0)$ for any odd map $g : S^n \to \mathbb{R}$. In the case of S^2, the first such algorithm was presented by Kulpa and Turzański in 2001 (see [16]). Recently, a similar algorithm, also for S^2, was obtained by Jayawant and Wong [14]. (The algorithm in [14] is a constructive proof of a theorem of Dyson [5], another classic result closely related to Borsuk–Ulam theorem. See also [22].)

Our main results, Theorems 3.2 and 5.3, are stated in more general terms for a unicoherent polyhedral complex (see Section 3). However, for the convenience of the reader, we include simple algorithms to approximate symmetric separators in I^n (Algorithm 4.1) and in S^n (Algorithm 6.1). The algorithms are implementation ready, written in a generic pseudo-code that can be easily translated into any real programming language.

Our results are based on the following idea. We begin with a symmetric space X that can be either S^n (with $n \geqslant 2$) or I^n or, more generally, a unicoherent polyhedral complex (see Section 3) invariant under a combinatorial involution T. Since X is locally connected and unicoherent we will be able to use arguments for separation similar to those in [7,11–13]. We consider a given symmetric triangulation $\mathcal{X}$ of X. (In the general case $\mathcal{X}$ is the T-symmetric partition defining the polyhedral complex.) We also consider a given odd (or T-odd) real function f defined on the set of vertices $\mathcal{X}$. We then construct a subcollection $\mathcal{C}$ of $\mathcal{X}$ in the following way. We start the construction by taking $\mathcal{C} = \{C_0\}$ where C_0 is some element of $\mathcal{X}$. In each step of the construction we enlarge $\mathcal{C}$ by adding to it all elements $C \in \mathcal{X}$ for which there is C' already in $\mathcal{C}$ such that either

- $C \cap C'$ contains a vertex v of $\mathcal{X}$ such that $f(v) = 0$, or
- $C \cap C'$ contains an edge $\langle u, v\rangle$ of $\mathcal{X}$ such that $f(u)f(v) < 0$.

We prove that if $\mathcal{C}$ is symmetric (or T-symmetric) then $\mathcal{C}^*$ is a symmetric (or T-symmetric) separator of X, see Theorem 3.2. This observation leads to an algorithm in the case of $X = I^n$ and more generally in the case where T has a fixed point $x_0 \in X_0$ ($x_0 = \mathbf{0}$ for $X = I^n$). In this case it is enough to start with C_0 containing x_0. In the case where there is no fixed point of T, we take a collection $\mathcal{L}$ of edges of $\mathcal{X}$ such that $\mathcal{L}^* \cup -\mathcal{L}^*$ (or $\mathcal{L}^* \cup T(\mathcal{L}^*)$) is a simple closed curve. We prove in Theorem 5.3 that exactly one collection $\mathcal{C}$ started with $C_0 \in \mathcal{L}$ is symmetric (T-symmetric), and therefore $\mathcal{C}^*$ is a symmetric (T-symmetric) separator of X.

If the function f (which is defined on the finite set of vertices of the triangulation $\mathcal{X}$) is a restriction of an odd (T-odd) map $g : X \to \mathbb{R}$, then each element of $\mathcal{C}$ intersects $g^{-1}(0)$. So, for sufficiently fine triangulation $\mathcal{X}$, $\mathcal{C}^*$ approximates a connected symmetric (T-symmetric) separator of X contained in $g^{-1}(0)$. The continuous map g and the underlying separator $g^{-1}(0)$, however, play no essential role in Theorems 3.2 and 5.3. The separation of X by $\mathcal{C}^*$ is purely combinatorial. It depends only on the information encoded in the finite function f. In our proof we observe that for each component B of the boundary of $\mathcal{C}^*$ in X, f has the same sign on all vertices of $\mathcal{X}$ contained in B. The separation of X by $\mathcal{C}^*$ follows from the following proposition.

Proposition 1.1. *Suppose A is a subcontinuum of a locally connected unicoherent continuum X and K is a component of $X \setminus A$. Then,* $\mathrm{cl}(K) \cap A$ *is connected.*

Proof. Since X is locally connected, K is open. It follows that $M = X \setminus K$ is a continuum and $\mathrm{cl}(K) \cap M = \mathrm{cl}(K) \cap A$. Since X is unicoherent and $M \cup \mathrm{cl}(K) = X$, the intersection of $\mathrm{cl}(K)$ and M is connected. □

2. Algorithm to find a symmetric separating arc in I^2

Any separator of S^n or I^n must be at least $(n-1)$-dimensional. Thus, if $n \geqslant 3$, a separator must be at least 2-dimensional. S^2 and I^2, however, are separated by 1-dimensional linear objects like simple closed curves or arcs (in case of I^2). This property makes the case of $n = 2$ unique, one can expect linear algorithms finding separators of S^2 and I^2. Several such algorithms, used in diverse contexts (see for example [8,9,14,16,20]), are based on the following observation. Assuming that each vertex of a triangle is colored with one of two colors, say black and white, then either the three vertices are colored with the same color, or there are exactly two edges with one vertex black and one vertex white. Suppose that each vertex of a certain triangulation of S^2 (or I^2) is painted either black or white. Suppose also that you are standing in a triangle having vertices of both colors. Now, walk through the triangulation, always entering the next triangle through a black-and-white edge and leaving it through the second such edge. On S^2, you will eventually return to the starting triangle, your path creating a separating cycle. On I^2, you may also wind up on the boundary. In that case backtrack to the starting triangle and continue further. You will eventually reach another point in the boundary, and your path is a separating arc joining two points in the boundary of I^2 through the interior of I^2. In this section we implement the above procedure in Algorithm 2.1 finding a symmetric separating arc in I^2. This algorithm is faster than the 2-dimensional version of more general Algorithm 4.1.

For any two points $a, b \in \mathbb{R}^2$, let $[a, b]$ denote the straight linear segment joining a and b.

Consider $\mathcal{T}$ a symmetric triangulation of I^2. (I.e. $\mathcal{T} = -\mathcal{T}$ is a finite collection of triangles such that $\mathcal{T}^* = I^2$, the intersection of any two distinct triangles in $\mathcal{T}$ is either empty, or a common vertex, or a common edge.) Let $\mathcal{V}$ be the set of vertices of $\mathcal{T}$. For any triangle $\Delta \in \mathcal{T}$ and two its distinct vertices u and v such that $[u, v]$ is not contained in the boundary of I^2, let $\mathrm{Nghbr}(\Delta, u, v)$ denote the triangle in $\mathcal{T} \setminus \{\Delta\}$ containing $[u, v]$.

Assume that $\mathbf{0} \notin \mathcal{V}$. Since $\mathcal{T}^* = I^2$, there is a triangle $\Delta_0 \in \mathcal{T}$ containing $\mathbf{0}$. Since $\Delta_0 \neq -\Delta_0$, $\mathbf{0}$ belongs to the interior of an edge of Δ_0. Let v_0 denote one vertex of this edge. Since this edge must be symmetric, it is equal to $[v_0, -v_0]$ for some vertex v_0. Observe that $[v_0, -v_0]$ is the common edge of Δ_0 and $-\Delta_0$. Also, if Δ and $-\Delta$ intersect for some $\Delta \in \mathcal{T}$, then Δ is either Δ_0 or $-\Delta_0$.

For our algorithm we need a set $P \subset \mathcal{V}$ such that P and $-P$ are disjoint and their union is $\mathcal{V}$. P should be defined in such a way that it is easy to check in the algorithm if $v \in P$. For example, one could define P as the set of those vertices $v = (x_1, x_2) \in \mathcal{V}$ that either $x_1 > 0$ or $x_1 = 0$ and $x_2 > 0$ Another way to define P could be to find a line $L \subset \mathbb{R}^2$ passing through $\mathbf{0}$ and missing $\mathcal{V}$. Now, P could be defined as the set of vertices of $\mathcal{V}$ on one side of L.

Finally, suppose that $f : \mathcal{V} \to \mathbb{R}$ is an odd function. We use f to define coloring of $\mathcal{V}$ with two colors. Let B be the set of those $v \in \mathcal{V}$ such that either $f(v) > 0$, or $f(v) = 0$ and $v \in P$. Set $W = -B$ Clearly, B and W are disjoint and their union is $\mathcal{V}$. We treat vertices in B as black and vertices in W as white.

The following algorithm finds a sequence $C_0, C_1, \ldots, C_j$ of triangles from $\mathcal{T}$ such that the following conditions are satisfied.

(1) $C_0 = \Delta_0$.
(2) For each $i = 1, \ldots, j$, the intersection of C_{i-1} and C_i is an edge with vertices a_i and b_i such that $f(a_i) \geqslant 0$ and $f(b_i) \leqslant 0$.
(3) C_j has a vertex w in the boundary of I^2.
(4) Let m_i denote the midpoint of $[a_i, b_i]$ for $i = 1, \ldots, j$. Additionally, let $m_0 = \mathbf{0}$ and $m_{j+1} = w$. Set $A = \bigcup_{i=0}^{j} [m_i, m_{i+1}]$ and $J = A \cup (-A)$. Then, J is a symmetric arc separating I^2 between x and $-x$ for each $x \in I^2 \setminus J$.

Algorithm 2.1.
Step 1: If either $f(v_0) > 0$ or $f(v_0) = 0$ and $v_0 \in P$ do Step 2, else do Step 3.
  Step 2: Set a to be v_0, and set b to be $-v_0$.
  Step 3: Set a to be $-v_0$, and set b to be v_0.
Step 4: Set C to be Δ_0 and add it to List $\mathcal{C}$.
Step 5: Set c to be the vertex of C other than a and b.
Step 6: While c does not belong to the boundary of I^2 do Steps 7–11.
  Step 7: If either $f(c) > 0$ or $f(c) = 0$ and $c \in P$ do Step 8, else do Step 9.
    Step 8: Set a to be c.
    Step 9: Set b to be c.
  Step 10: Set C to be $\mathrm{Nghbr}(C, a, b)$ and add it to List $\mathcal{C}$.
  Step 11: Set c to be the vertex of C other than a and b.
Step 12: Output List $\mathcal{C}$.

Proof. Observe that the algorithm enters the loop in Steps 6–11 with C_0 and $-C_0$ sharing vertices $a_0 \in B$ and $b_0 \in W$. Suppose that in the beginning of the ith pass through the loop the sequence $C_{i-1}, \ldots, C_0, -C_0, \ldots, -C_{i-1}$ contains $2i$ distinct triangles, every two consecutive of them sharing a common edge

$$[a_{i-1}, b_{i-1}], \ldots, [a_1, b_1], [a_0, b_0], -[a_1, b_1], \ldots, -[a_{i-1}, b_{i-1}], \tag{$*$}$$

respectively. One of the vertices of each of the edges $(*)$ belongs to B and the other belongs to W. It follows the edge $[a_i, b_i]$, chosen in Steps 7–9, is different from any of the edges $(*)$. Hence, C_i chosen in Step 10, is different from any of the $C_{i-1}, \ldots, C_0, -C_0, \ldots, -C_{i-1}$. Since C_i is neither $\Delta_0 = C_0$ nor $-\Delta_0$, it cannot intersect $-C_i$. So, after ith pass through the loop, we get the sequence $C_i, \ldots, C_0, -C_0, \ldots, -C_i$ that consists of $2(i+1)$ distinct triangles. Since $\mathcal{T}$ is finite, the construction has to end in Step 6 with the final sequence $C_j, \ldots, C_0, -C_0, \ldots, -C_j$ containing $2(j+1)$ distinct triangles. It follows that the conditions (1)–(3) are true and J is a symmetric arc intersecting $\mathrm{Bd}(I^2)$

at two points. Observe that J separates I^2 into two components K and K_1, one of which contains v_0 and the other $-v_0$. It follows that $K_1 = -K$, and the proof is complete. □

3. Odd functions on vertices of symmetric polyhedral complexes

Let m be a positive integer. If $a, b \in \mathbb{R}^m$, then by $\langle a, b\rangle$ we denote the straight linear segment in $\mathbb{R}^m$ between a and b. If $Z \subset \mathbb{R}^m$ by $A(Z)$ we denote the minimal affine subspace of $\mathbb{R}^m$ containing Z. A set $P \subset \mathbb{R}^m$ is *a convex polytope* (see [10, Chapter 16]) if there is a finite set V such that P is the convex hull of V. In this case we say that P is spanned by V. For a convex polytope P there is the unique minimal set of vertices $V(P)$ spanning P. If H is a hyperplane of $\mathbb{R}^m$ intersecting P but missing the interior of P in $A(P)$, the intersection $P \cap H$ is a (proper) face of P. Additionally, P itself is its own (improper) face. P has finitely many faces each of which is a convex polytope spanned by a subset of $V(P)$. One dimensional faces are called edges. Notice that each edge has exactly two vertices.

A polyhedral complex (see [10, p. 417 and p. 477]) is a finite non-empty collection of convex polytopes that contains all faces of all its elements, and such that the intersection of any two of its polytopes is either a common face of each of them or the empty set. Sometimes, a polyhedral complex is called a polytopial complex [10, p. 387]. If $\mathcal{X}$ is a polyhedral complex, then $\mathcal{X}^*$ is called the underlying polyhedron of $\mathcal{X}$. $\mathcal{V}(\mathcal{X})$ and $\mathcal{E}(\mathcal{X})$ denote the collections of vertices and edges, respectively. By *the generating collection* of $\mathcal{X}$ we understand the collection of elements of $\mathcal{X}$ that are maximal with respect to the inclusion. Clearly, $\mathcal{P} \subset \mathcal{X}$ is the generating collection of $\mathcal{X}$ if and only if $\mathcal{P}$ is minimal with respect to the property $\mathcal{P}^* = \mathcal{X}^*$.

Proposition 3.1. *Suppose $\mathcal{X}$ is a polyhedral complex, $X = \mathcal{X}^*$, $\mathcal{Z} \subset \mathcal{X}$ and B is the boundary of $\mathcal{Z}^*$ in X. Then there is $\mathcal{B} \subset \mathcal{X}$ such that $B = \mathcal{B}^*$. Suppose furthermore that C is a component of B. Then, C contains at least one vertex of $\mathcal{X}$. Also, if u and v are two different vertices of $\mathcal{X}$ contained in C, then there is $\mathcal{A} \subset \mathcal{E}(\mathcal{X})$ such that $\mathcal{A}^* \subset C$ is an arc with vertices u and v.*

Suppose that T is an involution on the underlying polyhedron of a polyhedral complex $\mathcal{X}$. We say that $\mathcal{X}$ is T-symmetric if for each $P \in \mathcal{X}$, $T(P) \in \mathcal{X}$ and T restricted to P is an isometry onto $T(P)$. Clearly, $T(\mathcal{V}(\mathcal{X})) = \mathcal{V}(\mathcal{X})$. Also, $T(\sum_{v\in V(P)} \lambda_v v) = \sum_{v\in V(P)} \lambda_v T(v)$ for each $\lambda_v \geqslant 0$ such that $\sum_{v\in V(P)} \lambda_v = 1$. If T is the antipodal map we say that $\mathcal{X}$ is symmetric.

If $\mathcal{X}$ is a T-symmetric polyhedral complex and f is a real function of $\mathcal{V}(\mathcal{X})$, we say that f is T-odd if $f(T(v)) = -f(v)$ for each $v \in \mathcal{V}(\mathcal{X})$. In such a case we denote by $\mathcal{E}_f$ the collection of those edges $e = \langle u, v\rangle \in \mathcal{E}(\mathcal{X})$ that $f(u)f(v) \leqslant 0$.

We say that the polyhedral complex $\mathcal{X}$ is connected if its underlying polyhedron $\mathcal{X}^*$ is connected. We say that $\mathcal{X}$ is unicoherent if $\mathcal{X}^*$ is unicoherent.

Theorem 3.2. *Suppose that $\mathcal{X}$ is a T-symmetric unicoherent connected polyhedral complex, and $f : \mathcal{V}(\mathcal{X}) \to \mathbb{R}$ is a T-odd function. Set $X = \mathcal{X}^*$. Let $\mathcal{C} \subset \mathcal{X}$ be such that $\mathcal{C}^*$ is a continuum invariant under T, and the boundary of $\mathcal{C}^*$ does not intersect $f^{-1}(0)$ and does not contain any edge from $\mathcal{E}_f$. Then, $\mathcal{C}^*$ separates X between x and $T(x)$ for each $x \in X \setminus \mathcal{C}^*$.*

Proof. Suppose that $\mathcal{C}^*$ does not separate X between x and $T(x)$ for some $x \in X \setminus \mathcal{C}^*$. Then there is a component K of $X \setminus \mathcal{C}^*$ such that $T(K) = K$. By Proposition 1.1, $B = \mathrm{cl}(K) \cap \mathcal{C}^*$ is a continuum. Clearly $T(B) = B$. By Proposition 3.1, there is a vertex $v \in \mathcal{V}(\mathcal{X})$ that belongs to B. Again by Proposition 3.1, there are vertices $v_0, v_1, \dots, v_j$ in $\mathcal{V}(\mathcal{X})$ such that $v_0 = v$, $v_j = T(v)$, and, for each $i = 1, \dots, j$, $\langle v_{i-1}, v_i\rangle \subset B$ is an edge of $\mathcal{X}$. Since $f(T(v)) = -f(v)$, there is $i = 1, \dots, j$ such that $f(x_{i-1})f(x_i) \leqslant 0$. This means that $\langle v_{i-1}, v_i\rangle \in \mathcal{E}_f$ is contained in the boundary of $\mathcal{C}^*$. This contradiction completes the proof. □

4. Algorithm to find a connected symmetric separator of I^n

Let k and n be positive integers. Consider the partition of I^n into k^n congruent closed n-dimensional cubes. Let $\mathcal{P}_k$ denote the collection of these cubes. Observe that the collection of the faces of all dimensions of cubes in $\mathcal{P}_k$ forms a symmetric polyhedral complex with $\mathcal{P}_k$ as its generating collection. Denote by $\mathcal{V}$ and $\mathcal{E}$ the sets of vertices and edges, respectively, of the complex. For any cube $C \in \mathcal{P}_k$ and its $(n-1)$-dimensional face F not contained in the boundary of I^n, let $\mathrm{Nghbr}(C, F)$ denote the cube in $\mathcal{P}_k \setminus \{C\}$ having F as its face.

Suppose $f : \mathcal{V} \to \mathbb{R}$ is such that $f(-v) = -f(v)$ for each $v \in \mathcal{V}$. Let $\mathcal{E}_f$ be the collection of those edges $e = \langle u, v\rangle \in \mathcal{E}$ that $f(u)f(v) \leqslant 0$.

The following algorithm finds a subcollection $\mathcal{C} \subset \mathcal{P}_k$ such that

(1) $-C \in \mathcal{C}$ for each $C \in \mathcal{C}$,
(2) each $C \in \mathcal{C}$ contains an edge from $\mathcal{E}_f$,
(3) $\mathcal{C}^*$ is connected, and
(4) $\mathcal{C}^*$ separates I^n between x and $-x$ for each $x \in I^n \setminus \mathcal{C}^*$.

Algorithm 4.1.

Step 1: Let C_1 a cube from $\mathcal{P}_k$ containing $\mathbf{0}$. Add C_1 to List A. Add C_1 to List C. If k is even add $-C_1$ to List C.
Step 2: While List A is not empty do Steps 3–8.
  Step 3: Set L to be any element in List A.
  Step 4: Remove L from List A.
  Step 5: For each $(n-1)$-dimensional face F of L such that $F \not\subset \mathrm{Bd}(I^n)$ and $\mathrm{Nghbr}(L, F)$ is not in List C do Steps 6–8.
    Step 6: If F contains an edge from $\mathcal{E}_f$, then do Steps 7 and 8.
      Step 7: Add $\mathrm{Nghbr}(L, F)$ to List A.
      Step 8: Add $\mathrm{Nghbr}(L, F)$ and $-\mathrm{Nghbr}(L, F)$ to List C.
Step 9: Output List C. (List C lists cubes in $\mathcal{C}$.)

Proof. List C is enlarged only in Steps 1 and 8, and a cube is added to the list always together with its symmetric twin. So $\mathcal{C}$ is symmetric after each step of the construction. Each cube added to List C shares a face with a cube previously added to the list. So $\mathcal{C}^*$ is connected after each step of the construction. Step 6 guarantees that no edge from $\mathcal{E}_f$ is in the boundary of $\mathcal{C}^*$. Since each vertex in $\mathrm{Bd}(\mathcal{C}^*)$ belongs to an edge also contained in $\mathrm{Bd}(\mathcal{C}^*)$, $\mathrm{Bd}(\mathcal{C}^*)$ does not intersect $f^{-1}(0)$. Now, the properties of the algorithm follow from Theorem 3.2. □

5. Finding connected symmetric separators in symmetric polyhedra

In the previous section, we were able to use Theorem 3.2 to find a symmetric separator in I^n because we could start Algorithm 4.1 from $\mathbf{0}$, the fixed point for the antipodal map on I^n. A similar procedure can be used for any T-symmetric polyhedral complex $\mathcal{X}$ starting from any fixed point of T. In this section we prove Theorem 5.3 allowing to find a T-symmetric separator starting from a T-invariant simple closed curve. This procedure is more complicated than the one used in Section 3, and it should be used only if T is fixed-point-free, or a fixed point of T is not readily available. In the case where T is the antipodal map on $S^n = \mathrm{Bd}(I^{n+1})$ subdivided into regular n-dimensional cubes, it is easy to directly find a symmetric simple closed curve (see Section 6). The following simple proposition may be helpful in the general case.

Proposition 5.1. *Suppose $\mathcal{X}$ is a T-symmetric polyhedral complex. Let $\mathcal{L}$ be a polyhedral complex that is contained in $\mathcal{V}(\mathcal{X}) \cup \mathcal{E}(\mathcal{X})$, and is minimal with respect to the properties $\mathcal{L}^*$ is connected and $\mathcal{L}^* \cap T(\mathcal{L}^*) \neq \emptyset$. Then, one of the following is true*

(1) *$\mathcal{L}$ is either a vertex or an edge invariant under T, or*
(2) *$\mathcal{L}^*$ and $T(\mathcal{L}^*)$ are two arcs intersecting at the common endpoints (hence, $\mathcal{L}^* \cup T(\mathcal{L}^*)$ is a simple closed curve).*

Suppose $\mathcal{P}$ is the generating collection of a T-symmetric polyhedral complex $\mathcal{X}$. Suppose also that $f : \mathcal{V}(\mathcal{X}) \to \mathbb{R}$ is a T-odd function. For any $e \in \mathcal{E}_f$, we will construct a sequence $\mathcal{C}_0(e), \mathcal{C}_1(e), \ldots$ by induction. Let $\mathcal{C}_0(e)$ be the collection of those $P \in \mathcal{P}$ that contain e. Suppose $\mathcal{C}_{i-1}(e)$ has been defined and define $\mathcal{C}_i(e)$ in the following way. Let $\mathcal{C}_i(e)$ be the collection of those $P \in \mathcal{P}$ such that the intersection $P \cap \mathcal{C}_{i-1}(e)^*$ either contains an edge from $\mathcal{E}_f$ or a vertex from $f^{-1}(0)$. Clearly, $\mathcal{C}_{i-1}(e) \subset \mathcal{C}_i(e)$. Since $\mathcal{P}$ is finite, there is an integer $q \geqslant 0$ such that $\mathcal{C}_q(e) = \mathcal{C}_{q+1}(e)$.

Let $q(e)$ be the first such number. Set $\mathcal{C}(e) = \mathcal{C}_{q(e)}(e)$. Notice that $\mathcal{C}(e)$ is the subcollection of $\mathcal{P}$ minimal with respect to the following properties:

(C-1) $e \subset \mathcal{C}(e)^*$,
(C-2) $\mathcal{C}(e)^*$ is connected,
(C-3) the boundary of $\mathcal{C}(e)^*$ does not intersect $f^{-1}(0)$, and
(C-4) for each component B of $\mathrm{Bd}(\mathcal{C}(e)^*)$, f has the same sign on all vertices of $\mathcal{X}$ contained in B.

Proposition 5.2. *Suppose $e, d \in \mathcal{E}_f$. Then the following properties are true.*

(1) $\mathcal{C}(T(e)) = T(\mathcal{C}(e))$.
(2) $d \in \mathcal{C}(e)^*$ *if and only if there is a sequence $P_0, P_1, \dots, P_j$ of elements of $\mathcal{P}$ such that $e \subset P_0$, $d \subset P_j$, and $P_{i-1} \cap P_i$ contains either a vertex from $f^{-1}(0)$ or an edge from $\mathcal{E}_f$ for each $i = 1, \dots, j$.*
(3) *Either $\mathcal{C}(d) \cap \mathcal{C}(e) = \emptyset$ or $\mathcal{C}(d) = \mathcal{C}(e)$.*

Theorem 5.3. *Suppose $\mathcal{X}$ is a T-symmetric polyhedral complex such that $X = \mathcal{X}^*$ is connected and unicoherent. Suppose also that $f : \mathcal{V}(\mathcal{X}) \to \mathbb{R}$ is a T-odd function. Let $\mathcal{L}$ be a subcollection of $\mathcal{E}(\mathcal{X})$ such that $\mathcal{L}^*$ and $T(\mathcal{L}^*)$ are two arcs intersecting at their common endpoints s and $T(s)$. Then there is $d \in \mathcal{L} \cap \mathcal{E}_f$ such that*

(1) $T(\mathcal{C}(d)^*) = \mathcal{C}(d)^*$,
(2) *$\mathcal{C}(d)^*$ separates X between x and $T(x)$ for each $x \in X \setminus \mathcal{C}(d)^*$, and*
(3) *for any $d' \in \mathcal{L} \cap \mathcal{E}_f$, if $T(d') \subset \mathcal{C}(d')^*$ then $\mathcal{C}(d') = \mathcal{C}(d)$.*

Proof. For each $F \in \mathcal{X}$, let $y[F] = \frac{1}{\ell} \sum_{v \in V(F)} v$ where ℓ denotes the number of vertices in $V(F)$. Clearly, $T(y[F]) = y[T(F)]$.

Let $\mathcal{P}$ be the generating collection $\mathcal{X}$. For each $P \in \mathcal{P}$, let $Y[P]$ be the union of segments $\langle y[P], y[F] \rangle$ where $F \subset \mathrm{Bd}(P)$ and F is either an edge from $\mathcal{E}_f$ or a vertex from $f^{-1}(0)$. Observe that $T(Y[P]) = Y[T(P)]$.

Let Y denote the union $\bigcup_{P \in \mathcal{P}} Y[P]$. Let $\mathcal{D}$ be the subcollection of those $F \in \mathcal{X}$ that $F \cap f^{-1}(0) = \emptyset$ and F does not contain any edge from $\mathcal{E}_f$. There is a positive number δ such that the distance between Y and any $F \in \mathcal{D}$ is greater than δ. Let η be such that $0 < \eta < \delta$ and if F_1 and F_2 are two nonintersecting elements of $\mathcal{X}$, then the distance between F_1 and F_2 is greater than 2η. Clearly, $T(F) \in \mathcal{D}$ for each $F \in \mathcal{D}$. Since T restricted to P is an isometry onto $T(P)$ for each $P \in \mathcal{P}$, it follows that $T(W(F)) = W(T(F))$ for each $F \in \mathcal{D}$. Also, observe the following claim.

Claim 5.3.1. *Suppose $P \in \mathcal{P}$ and $\mathcal{B} \subset \mathcal{D}$. Let G denote $\bigcup_{F \in \mathcal{B}} W(F)$. Then, $Y[P] \cap G = \emptyset$ and $P \setminus G$ is connected.*

Denote $\mathcal{L}^* \cup T(\mathcal{L}^*)$ by S. Observe that S is a simple closed curve such that $T(S) = S$. Let $\mathcal{K} = \{\mathcal{C}(e)^* \mid e \in (\mathcal{L} \cup T(\mathcal{L})) \cap \mathcal{E}_f\}$. Let s be an endpoint of $\mathcal{L}^*$. Since both $s, T(s) \in \mathcal{L}^*$, $\mathcal{L} \cap \mathcal{E}_f \neq \emptyset$. Thus, $\mathcal{K} \neq \emptyset$. By Proposition 5.2, $T(K) \in \mathcal{K}$ for each $K \in \mathcal{K}$.

For each $K \in \mathcal{K}$, let $\mathcal{B}(K)$ be the collection of the components of the boundary of K. Clearly, each $B \in \mathcal{B}(K)$ is the union of a subcollection of $\mathcal{X}$. It follows from (C-3) and (C-4) that either f is positive on all vertices of $\mathcal{V}(\mathcal{X})$ belonging to B, or f is negative on those vertices. We will say that B is f-positive in the first case, and f-negative, otherwise. It follows, in particular, that B is the union of a subcollection of $\mathcal{D}$.

For each $K \in \mathcal{K}$ and each $B \in \mathcal{B}(K)$, let (B, K) denote the union of the sets $W(F) \cap K$ where $F \subset B$ is in $\mathcal{D}$. Let

$$\mathcal{Z} = \{(B, K) \mid K \in \mathcal{K} \text{ and } B \in \mathcal{B}(K)\}.$$

The following claim is an easy consequence of the definition of $\mathcal{Z}$.

Claim 5.3.2. *Suppose (B', K') and (B'', K'') are two distinct elements of $\mathcal{Z}$. Then, $(B', K') \cap (B'', K'') = B' \cap B''$.*

For each $K \in \mathcal{K}$, let $\widetilde{K}$ denote the closure of $K \setminus \bigcup_{B \in \mathcal{B}(K)} (B, K)$, and let $\mathcal{J}(K) = \{(B, K) \cap \widetilde{K} \mid B \in \mathcal{B}(K)\}$. Set $\mathcal{J} = \bigcup_{K \in \mathcal{K}} \mathcal{J}(K)$ and $\widetilde{\mathcal{K}} = \{\widetilde{K} \mid K \in \mathcal{K}\}$. It follows from Claim 5.3.2 that elements of $\mathcal{J}$ are mutually disjoint.

Clearly, $\mathcal{J}(K)^*$ is the boundary $\widetilde{K}$ in X for each $K \in \mathcal{K}$. Also, $\widetilde{\mathcal{K}}^* = T(\widetilde{\mathcal{K}}^*)$, elements of $\widetilde{\mathcal{K}}$ are mutually disjoint, and $\mathrm{Bd}(\widetilde{\mathcal{K}}^*) = \mathcal{J}^*$.

Claim 5.3.3. *For each $K \in \mathcal{K}$, $\widetilde{K}$ is a continuum intersecting S.*

Proof of 5.3.3. Let $K = \mathcal{C}(e)^*$ for some $e \in \mathcal{E}_f$. Using induction, observe that $Y \cap \mathcal{C}_i(e)^*$ is connected for $i = 0, \ldots, q(e)$. Hence, $Y \cap \mathcal{C}(e)^*$ is connected. Now, the claim follows from 5.3.1. $\square$

It follows from 5.3.3 that $S \cup \widetilde{\mathcal{K}}^*$ is a continuum.

Let J_+ denote the union of those $(B, K) \cap \widetilde{K} \in \mathcal{J}$ for which B is f-positive. Similarly, let J_- be the union of those $(B, K) \cap \widetilde{K} \in \mathcal{J}$ for which B is f-negative. Observe that J_+ and J_- are closed sets such that $J_+ \cap J_- = \emptyset$, $J_+ \cup J_- = \mathcal{J}^*$ and $J_- = T(J_+)$.

Claim 5.3.4. *Suppose M is a component of $S \setminus \widetilde{\mathcal{K}}^*$. Let z' and z'' denote the endpoints of the closure of M. Then, the following conditions are true.*

(1) *Either both z' and z'' belong to J_+ or they both belong to J_-.*
(2) *If D is a component of $X \setminus (S \cup \widetilde{\mathcal{K}}^*)$ such that $\mathrm{cl}(D) \cap M \neq \emptyset$ and z' and z'' are not in $\mathrm{cl}(D)$, then $\mathrm{cl}(D) \cap (S \cup \widetilde{\mathcal{K}}^*) \subset M$. In particular, it follows that in this case $\mathrm{cl}(D) \cap \mathcal{J}^* = \emptyset$.*

Proof of 5.3.4. The collection $\mathcal{L}$ contains an edge $e_0 \in \mathcal{E}_f$. Since $T(e_0) \neq e_0$ and $y[e_0]$, $y[T(e_0)]$ belong to the interior of $\widetilde{\mathcal{K}}^*$, $y[e_0]$, $y[T(e_0)]$ do not belong to M. It follows that z' and z'' are not contained in the same edge.

Let $(B', K'), (B'', K'') \in \mathcal{Z}$ be such that $z' \in (B', K') \cap \widetilde{K'}$ and $z'' \in (B'', K'') \cap \widetilde{K''}$. Let e' and e'' denote the edges in $\mathcal{L} \cup T(\mathcal{L})$ containing z' and z'', respectively. Observe that the intersection $e' \cap B'$ consists of exactly one vertex of $\mathcal{X}$. Denote this by v'. Observe that $v' \in M$. Similarly, let $v'' \in M$ be the vertex of $\mathcal{X}$ that is the intersection $e'' \cap B''$. Suppose one of the numbers $f(v')$ and $f(v'')$ is positive while the other is negative. Then, M must contain an edge $e \in \mathcal{E}_f$. Observe that $y[e]$ must belong to $\widetilde{\mathcal{K}}^*$, a contradiction. Thus, (1) holds.

To prove (2), suppose D is a component of $X \setminus (S \cup \widetilde{\mathcal{K}}^*)$ such that $\mathrm{cl}(D) \cap M \neq \emptyset$ and z' and z'' are not in $\mathrm{cl}(D)$. By 1.1, $\mathrm{cl}(D) \cap (S \cup \widetilde{\mathcal{K}}^*)$ is connected. As $M \setminus \{z', z''\}$ is a component of $S \cup \widetilde{\mathcal{K}}^* \setminus \{z', z''\}$, $\mathrm{cl}(D) \cap (S \cup \widetilde{\mathcal{K}}^*) \subset M \setminus \{z', z''\}$. $\square$

We will now prove the following two claims.

Claim 5.3.5. *For each component D of $X \setminus (S \cup \widetilde{\mathcal{K}}^*)$, either $\mathrm{cl}(D) \cap J_+ = \emptyset$ or $\mathrm{cl}(D) \cap J_- = \emptyset$.*

Proof of 5.3.5. Suppose $\mathrm{cl}(D)$ intersects some $J', J'' \in \mathcal{J}$. Since J' and J'' are contained in $\widetilde{\mathcal{K}}^*$, they do not intersect D and, therefore, $\mathrm{Bd}(D)$ intersects both of them. By Proposition 1.1, $\mathrm{Bd}(D)$ is connected. As $\mathrm{Bd}(\widetilde{\mathcal{K}}^*) = \mathcal{J}^*$, $\mathrm{Bd}(D)$ is contained in $(S \setminus \widetilde{\mathcal{K}}^*) \cup \mathcal{J}^*$. Since $\mathcal{J}$ is finite and its elements are closed and mutually disjoint, there is a finite sequence $J_0, J_1, \ldots, J_j$ of elements of $\mathcal{J}$ such that $J_0 = J'$, $J_j = J''$, and, for each $i = 1, \ldots, j$, J_{i-1} and J_i are intersected by the closure of one component of $S \setminus \widetilde{\mathcal{K}}^*$. Now, the claim follows from Claim 5.3.4 (1). $\square$

Claim 5.3.6. *For each component E of $X \setminus \widetilde{\mathcal{K}}^*$, either $\mathrm{cl}(E) \cap J_+ = \emptyset$ or $\mathrm{cl}(E) \cap J_- = \emptyset$.*

Proof of 5.3.6. Take any $a, b \in \mathrm{cl}(E) \cap \mathcal{J}^*$. There is a sequence $D_0, D_1, \ldots, D_j$ of components of $E \setminus S$ such that $a \in \mathrm{cl}(D_0)$, $b \in \mathrm{cl}(D_j)$, and $\mathrm{cl}(D_{i-1}) \cap \mathrm{cl}(D_i) \cap (S \setminus \widetilde{\mathcal{K}}^*) \neq \emptyset$ for each $i = 1, \ldots, j$. Let $i(0), i(1), \ldots, i(k)$ be integers such that $0 \leqslant i(0) < i(1) < \cdots < i(k) \leqslant j$ and, for each $i = 0, \ldots, j$, $\mathrm{cl}(D_i) \cap \mathcal{J}^* \neq \emptyset$ if and only if $i = i(p)$ for some $p = 0, \ldots, k$. Clearly, $i(0) = 0$ and $i(k) = j$.

For each $p = 0, \ldots, k-1$, let M_p denote a component of $S \setminus \widetilde{\mathcal{K}}^*$ intersecting $\mathrm{cl}(D_{i(p)}) \cap \mathrm{cl}(D_{i(p)+1})$. Denote the endpoints of M_p by z'_p and z''_p. By 5.3.4 (2), $\mathrm{cl}(D_{i(p)}) \cap \{z'_p, z''_p\} \neq \emptyset$. Let b_p be one of the points z'_p and z''_p that belongs to $\mathrm{cl}(D_{i(p)})$. We will now observe that

$$\mathrm{cl}(D_{i(p+1)}) \cap M_p \neq \emptyset. \tag{$*$}$$

If $i(p+1) = i(p) + 1$, then $(*)$ is obvious. Suppose that $i(p+1) > i(p) + 1$. We will prove that

$$\mathrm{cl}(D_i) \cap (S \cup \widetilde{\mathcal{K}}^*) \subset M_p \setminus \{z'_p, z''_p\} \quad \text{for } i = i(p)+1, \ldots, i(p+1)-1. \tag{$**$}$$

Since $\mathrm{cl}(D_{i(p)+1})$ intersects M_p, we get $(**)$ for $i = i(p)+1$ by using 5.3.4 (2). It follows that $\mathrm{cl}(D_{i(p)+2})$ intersects M_p. If $i(p)+2 < i(p+1)$, we use 5.3.4 (2) again to get $(**)$ for $i = i(p)+2$. Consequently, $\mathrm{cl}(D_{i(p)+3}) \cap M_p \neq \emptyset$. Either $i(p)+3 = i(p+1)$ or we continue to use 5.3.4 (2) repeatedly until we prove $(**)$ for $i = i(p)+1, \ldots, i(p+1)-1$. Now, $(*)$ follows from $(**)$ for $i = i(p+1)-1$.

Since $\mathrm{cl}(D_{i(p+1)}) \cap \mathcal{J}^* \neq \emptyset$, it follows from $(*)$ and 5.3.4 (2) that $\mathrm{cl}(D_{i(p+1)}) \cap \{z'_p, z''_p\} \neq \emptyset$. Let a_{p+1} be one of the points z'_p and z''_p that belongs to $\mathrm{cl}(D_{i(p+1)})$. So, we have defined $b_0, \ldots, b_{k-1}$ and $a_1, \ldots, a_k$. Additionally, set $a_0 = a$ and $b_k = b$.

For each $p = 0, \ldots, k$, $a_p, b_p \in \mathrm{cl}(D_{i(p)})$, so by 5.3.5, either both a_p and b_p are in J_+ or they both are in J_-. For each $p = 0, \ldots, k-1$, $b_p, a_{p+1} \in \{z'_p, z''_p\}$, so by 5.3.4 (1), either both b_p and a_{p+1} are in J_+ or they both are in J_-. Accordingly, either all $a_0, \ldots, a_k, b_0, \ldots, b_k$ are in J_+ or all those points are in J_-. Therefore, either both $a = a_0$ and $b = b_k$ are in J_+ or they both are in J_-. □

Since $T(\widetilde{\mathcal{K}}^*) = \widetilde{\mathcal{K}}^*$, $J_- = T(J_+)$ and the closure of each component E of $X \setminus \widetilde{\mathcal{K}}^*$ intersects $J_+ \cup J_-$, it follows from 5.3.6 that $\widetilde{\mathcal{K}}^*$ separates X between x and $T(x)$ for each $x \in X \setminus \widetilde{\mathcal{K}}^*$. By [13, Theorem 3], $\widetilde{\mathcal{K}}^*$ has component H such that $H = T(H)$ and H separates X between x and $T(x)$ for each $x \in X \setminus H$. Since elements of $\widetilde{\mathcal{K}}$ are mutually disjoint continua, there is $K \in \mathcal{K}$ such that $\widetilde{K} = H$. As $\widetilde{K} \subset K$, it follows that $T(K) = K$ and K separates X between x and $T(x)$ for each $x \in X \setminus K$. By the definition of $\mathcal{K}$, there is $e \in (\mathcal{L} \cup T(\mathcal{L})) \cap \mathcal{E}_f$ such that $K = \mathcal{C}(e)^*$. Now, define $d = e$ if $e \in \mathcal{L}$ or set $d = T(e)$ if $e \in T(\mathcal{L})$. Since

$$\mathcal{C}\big(T(e)\big)^* = T\big(\mathcal{C}(e)^*\big) = T(K) = K,$$

the proof of (1) and (2) in the statement of the theorem is complete.

Suppose that there is $d' \in \mathcal{L} \cap \mathcal{E}_f$ such that $T(d') \subset \mathcal{C}(d')^*$ and $\mathcal{C}(d') \neq \mathcal{C}(d)$. Set $K' = \mathcal{C}(d')^*$. Now, $\widetilde{K'}$ contains both $y[d']$ and $T(y[d']) = y[T(d')]$. Since $\widetilde{K'}$ is connected and $\widetilde{K'} \cap K = \emptyset$, K does not separate X between $y[d']$ and $T(y[d'])$, a contradiction proving (3) in the statement of the theorem. □

6. Algorithm to find a connected symmetric separator of S^n for $n \geqslant 2$

Let k be an even positive integer. Consider the partition of $S^n = \mathrm{Bd}(I^{n+1})$ resulting from partitioning each of the n-dimensional faces of I^{n+1} into k^n congruent closed n-dimensional cubes. Let $\mathcal{S}_k$ denote the collection of these cubes. Observe that the collection of the faces of all dimensions of cubes in $\mathcal{S}_k$ forms a symmetric polyhedral complex with $\mathcal{S}_k$ as its generating collection. Denote by $\mathcal{V}$ and $\mathcal{E}$ the sets of vertices and edges, respectively, of the complex.

For each cube in $C \in \mathcal{S}_k$, let $\mathrm{Edges}(C)$ denote the collection of edges of C. For each edge $e \in \mathcal{E}$, let $\mathrm{Cubes}(e)$ denote the collection of those cubes in $\mathcal{S}_k$ that contain e.

Suppose $f : \mathcal{V} \to \mathbb{R}$ is such that $f(-v) = -f(v)$ for each $v \in \mathcal{V}$. Let $\mathcal{E}_f$ be the collection of those edges $e = \langle u, v\rangle \in \mathcal{E}$ that $f(u)f(v) \leqslant 0$.

We will now use Theorem 5.3 to obtain an algorithm to find $\mathcal{C}(d) \subset \mathcal{S}_k$ for some $d \in \mathcal{E}_f$ such that

(1) $-C \in \mathcal{C}(d)$ for each $C \in \mathcal{C}(d)$,
(2) each $C \in \mathcal{C}(d)$ contains an edge from $\mathcal{E}_f$,
(3) $\mathcal{C}(d)^*$ is connected, and
(4) $\mathcal{C}(d)^*$ separates S^n between x and $-x$ for each $x \in S^n \setminus \mathcal{C}(d)^*$.

In order to use 5.3, we must supply a suitable $\mathcal{L}$. (For this purpose, it was convenient to assume that k is even.) For each $i = 0, \ldots, n$, let $p_i = (x_1, x_1, \ldots, x_n, 0) \in \mathrm{Bd}(I^{n+1})$ be such that $x_j = 1$ if $1 \leqslant j \leqslant i$ and $x_j = -1$ if $i < j \leqslant n$. Set $L = \bigcup_{i=1}^n \langle p_i, p_{i-1}\rangle$ and $S = L \cup (-L)$. Observe that L is an arc and S is a symmetric simple closed curve contained in S^n. Since k is even, the arc L is the union of nk edges from $\mathcal{E}$. Let $\mathcal{L}$ denote the collection of those nk edges.

Algorithm 6.1.

Step 1: Add all edges in $\mathcal{L} \cap \mathcal{E}_f$ to List D.
Step 2: While List D is not empty do Steps 3–15.
  Step 3: Set d to be any element of List D.
  Step 4: Remove d from List D, and add d to list A.
  Step 5: While List A is not empty do Steps 6–13.
    Step 6: Set a to be any element of List A.
    Step 7: Remove a from List A, and add a to List B.
    Step 8: For each cube c in Cubes(a), if c is not in List C do Steps 9–13.
      Step 9: Add c to List C.
      Step 10: For each edge e in Edges(c), if e is in $\mathcal{E}_f$ but not in List A and not in List B, then do Steps 11–13.
        Step 11: Add e to List A.
        Step 12: If e belongs to List D, remove it from this list.
        Step 13: If $e = -d$, set SuccessFlag to TRUE.
  Step 14: If SuccessFlag is TRUE, then output List C and stop the program. (List C lists cubes in $\mathcal{C}(d)$.)
  Step 15: Remove all elements from Lists B and C.

Proof. The algorithm enters Step 6 with list C empty and list A containing only one edge $d \in \mathcal{L} \cap \mathcal{E}_f$. Then all cubes containing d are added to list C. If a cube is in list C then all its edges that are in $\mathcal{E}_f$ are added to list A and eventually all cubes containing these edges will be added to list C. Denote by $\mathcal{C}$ the union of the cubes in list C at the moment when the program enters Step 14. Clearly, $\mathcal{C}$ is a minimal collection satisfying the following properties: $d \subset \mathcal{C}^*$, $\mathcal{C}^*$ is connected and does not contain edges from $\mathcal{E}_f$ in its boundary. To prove that $\mathcal{C} = \mathcal{C}(d)$ we need to show that $f^{-1}(0)$ does not intersect the boundary of $\mathcal{C}^*$. Suppose to the contrary that there is a vertex $v \in \mathrm{Bd}(\mathcal{C}^*)$ such that $f(v) = 0$. Since $\mathrm{Bd}(\mathcal{C}^*)$ does not contain isolated points, v must be an endpoint of some edge $e \subset \mathrm{Bd}(\mathcal{C}^*)$. Observe that $e \in \mathcal{E}_f$, a contradiction proving that $\mathcal{C} = \mathcal{C}(d)$.

Finally, notice that, by Theorem 5.3(3), Step 13 guaranties that the algorithm exits in Step 14 when list C contains the separator. □

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